BIVARIATE POLYNOMIALS OF LOW DEGREE AND SMALL MAHLER MEASURE

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1. INTRODUCTION

Definition of the Mahler measure of an algebraic integer

In 1962, K. Mahler defined the measure that would later bear his name in (Mahler 1962). Let $P$ be a polynomial with complex coefficients. We write $P$ in the form

$$P(x) = a_0 \prod_{i=1}^{n} (x - \alpha_i),$$

where $a_0$ is the leading coefficient of $P$ and the $\alpha_i$’s are its complex roots. The Mahler measure of $P$ is defined as

$$M(P) = |a_0| \prod_{|\alpha_i| \geq 1} |\alpha_i|.$$

The Mahler measure of a polynomial with integer coefficients, being the product of the moduli of complex numbers outside the unit circle and the modulus of a non-zero integer, is therefore a strictly positive real number greater than 1. The natural logarithm of the Mahler measure is also of significant importance and is denoted by

$$m(P) = \log(M(P)).$$

The measure of an algebraic number $\alpha$ is defined by

$$M(\alpha) = M(P_\alpha),$$

where $P_\alpha$ denotes the minimal polynomial of $\alpha$.

Similarly, we define $m(\alpha) = \log(M(\alpha))$.

Properties of the Mahler measure in one dimension

Theorem 1 (Kronecker’s Theorem). If $p$ is an irreducible monic polynomial with integer coefficients and with $M(p) = 1$, then either $p(z) = z$, or $p$ is a cyclotomic polynomial.

C. Smyth proves the following result (Smyth 1972):

Theorem 2. If $P \in \mathbb{Z}[x]$ is monic, irreducible, and non-reciprocal, then $M(P) \geq M(x^3 - x - 1) = \theta \approx 1.324717$.

Dobrowolski showed the following important lower bound is presented (Dobrowolski 1972):

Theorem 3. If $P \in \mathbb{Z}[x]$ is monic, irreducible, and non-cyclotomic of degree $d$, then $M(P) \geq 1 + c \left( \frac{\log \log d}{\log d} \right)^3$, where $c$ is an absolute positive constant.

Let us also recall a result due to A. Schinzel (Schinzel 1973):

Theorem 4. If $P \in \mathbb{Z}[x]$ is monic, of degree $d$, has all real roots, and satisfies $P(1)P(-1) \neq 0$ and $|P(0)| = 1$, then $M(P) \geq \left( \frac{1+\sqrt{5}}{2} \right)^{d/2}$.
V. Flammang extended Schinzel’s theorem (Flammang 1997) by proving the following result:

**Theorem 5.** Any monic $P \in \mathbb{R}[x]$ of degree $d$ whose all zeros are real (resp. positive) and such that $P(0) \neq 0$, $P(1) \lor 1$, $P(-1) \lor 1$ satisfies $2M(P)^d \geq 1 + \left(4|P(0)|^\frac{2}{d} + 1\right)^\frac{1}{2}$ (resp. $2M(P)^d \geq 1 + \left(4|P(0)|^\frac{1}{d} + 1\right)^\frac{1}{2}$).

This result is generalized in (Ounaies et.al 2021), which includes the following theorem:

**Theorem 6.** Let $P \in \mathbb{C}[x]$ be a monic polynomial of degree $d \geq 2$. Assume that $P$ has $m \geq 1$ real zeros and that $P(0)P(1)P(-1) \neq 0$. Then $2^\frac{d}{m}M(P)^\frac{2}{m} \geq |P(1)P(-1)|^\frac{1}{m} + \left(\frac{d}{4^m}P(0)^\frac{2}{m} + |P(1)P(-1)|^\frac{2}{m}\right)^\frac{1}{2}$.

**Remark.** A real algebraic integer greater than 1 is a Salem number if all its conjugates have a modulus less than or equal to 1, and if at least one conjugate has a modulus equal to 1. The minimal polynomials of Salem numbers are actually interesting candidates for finding small Mahler measures of univariate polynomials, as their Mahler measure is often low. This property is crucial for identifying polynomials that minimize the Mahler measure in one dimension. In the context of this work, we are interested in the Mahler measure of bivariate polynomials.

**Lehmer’s Conjecture**

The Lehmer’s Conjecture is the question of whether one can approach 1 as closely as desired by Mahler measures of algebraic integers. This question was posed by Lehmer in 1933 in (Lehmer 1933). Mathematicians believe the answer is negative. Thus, Lehmer’s conjecture is formulated as follows: there exists a constant $c > 0$ such that for any algebraic number $\alpha$ with $M(\alpha) > 1$, we have $M(\alpha) \geq 1 + c$.

Various approaches have been initiated, and although the conjecture remains unresolved, several results have been established and the conjecture has been resolved in specific cases. In the next subsection, we show that the conjecture is true for algebraic integers whose minimal polynomial is not reciprocal. Therefore, to approach 1, one must do so via algebraic integers with reciprocal minimal polynomials, with Salem numbers being candidates for this.

The smallest known Mahler measure strictly greater than 1 is that of the reciprocal polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. It was found by Lehmer and is approximately 1.176. This is also the smallest known Salem number, called Lehmer’s number, and the polynomial is referred to as Lehmer’s polynomial.

**Smyth’s Theorem for Non-reciprocal Algebraic Numbers**

We remind the following theorem:

**Theorem 7.** If $P \in \mathbb{Z}[x]$ is not self-reciprocal and satisfies $P(0)P(1) \neq 0$, then $M(P) \geq \theta_0$, where $\theta_0 = 1.324717 \ldots$ is the real root of the polynomial $x^3 - x - 1$. 

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This result is due to C. Smyth and was published in (Smyth 1971) The number \( \theta_0 \) was already known from the work of C. Siegel (see Siegel 1944) as the smallest Pisot number.

2. MAHLER MEASURES IN SEVERAL VARIABLES

Definition of the Mahler measure of multivariate polynomials

Here we generalize the Mahler measure of a single-variable polynomial to multivariable polynomials. We will then focus more specifically on two-variable polynomials. To do so, we provide another expression for the Mahler measure of a polynomial. The following lemma is based on an equality proven by Jensen, found in (Jensen 1899).

**Lemma 2.1.** Let \( P \in \mathbb{C}[X] \) be a polynomial. Then:

\[
M(P) = \exp \left( \int_0^1 \log |P(e^{2\pi i t})| \, dt \right)
\]

or equivalently:

\[
M(P) = \exp \left( \int_T \log |P| \, d\mu \right).
\]

This expression is generalized to polynomials in several variables.

**Definition 2.1.** Let \( P \in \mathbb{C}[X_1, \ldots, X_n] \) be a polynomial in \( n \) variables. The Mahler measure of \( P \) is defined as:

\[
M(P) = \exp \left( \int_{\mathbb{T}^n} \log |P| \, d\mu \right).
\]

As we will see, Mahler measures in several variables are, in a sense, limit points of Mahler measures in one variable. This is the subject of the next subsection.

Connection between Mahler’s Measures of Univariate and Bivariate Polynomials

As explained in (Boyd 2005), it was proposed in the paper (Boyd 1981) that the minimal limit points of the Mahler measure in one dimension could be determined by evaluating the measures of polynomials with two variables, and suggestions were made regarding the smallest two limit points. In the literature, the stated aim is then to draw up lists of limit points for the set of Mahler measures of univariate polynomials. Usually, we look for values below 1.37. To prove that a real number is a limit point of the set of Mahler measures of univariate polynomials, we need to show that this real number is the Mahler measure of a bivariate polynomial. The largest list available can be found in (Boyd 2005). The articles (. El Otmani et. al 2019) and (El Otmani et. al 2021) add new limit points to the aforementioned list.

In this paper, our goal is to demonstrate, for a substantial number of known limit points, that each of these limit points is the Mahler measure of a bivariate polynomial whose degree is significantly lower than those of the polynomials initially provided in the articles containing the aforementioned lists.
Our method will involve employing the Taboo method to minimize the Mahler measure of bivariate
hexanomials with a maximum degree of 3, then a maximum degree of 4, and finally a maximum degree
of 5 (in both x and y). In the following section, we provide a brief overview of the principle of the
Taboo method.

3. ALGORITHMIC APPROACH

Overview of the Taboo Method

The Taboo search method, originally proposed by Glover in 1986, is an advanced optimization
technique used to navigate the search space of potential solutions in combinatorial problems. Unlike
simple descent methods, which stop at local minima, the Taboo method allows for temporary
degradation of the solution to escape local optima and explore new regions of the solution space. Here
is a brief presentation of the Taboo method based on the application to binary constraint problems:

Principle of the Taboo Method

The core idea of the Taboo method is to avoid cycling back to previously visited solutions by
maintaining a list of forbidden moves, known as the Taboo list. This list records certain attributes of
the solutions that have been explored, preventing the algorithm from revisiting them for a defined
period, known as the Taboo tenure.

Implementation Steps

1. Initialization: Start with an initial solution and calculate its objective function value. Initialize
the Taboo list to be empty.

2. Neighborhood Search: At each iteration, examine the neighboring solutions of the current
solution. A neighbor is typically defined as a solution that can be reached by a simple alteration, such
as changing the value of one variable.

3. Taboo List Management: - If a move is in the Taboo list, it is usually forbidden, unless it meets
an aspiration criterion (i.e., if it produces a solution better than any seen so far). - Add the inverse of
the current move to the Taboo list, ensuring that this move cannot be undone in the next few iterations.

4. Aspiration Criterion: Allow moves that are in the Taboo list if they lead to a solution better
than the best-known solution.

5. Stopping Condition: Continue iterating until a predetermined number of iterations is reached
or no improvement is found for a set number of iterations.

Pseudo-Algorithm

Initialize the current solution S
Initialize the best solution S* and best objective value f*
Initialize Taboo list T

While stopping criteria not met:
Generate neighborhood N(S) of current solution S
Select the best neighbor S’ from N(S) that is not in Taboo list T or meets aspiration criteria
Update current solution S <- S’
If objective value of S is better than f*:
    Update best solution S* <- S
    Update best objective value f* <- f(S)
Update Taboo list T with inverse of the move made to reach S’

Return S*, f*
The implementation of the Taboo method was carried out in MATLAB language.

4. NUMERICAL RESULTS

In the following table, we compile a number of known limit points along with their original bivariate polynomials and the low-degree bivariate polynomials found by our method. When the reference polynomial is already of low degree, it is left unchanged. We use the notations from (Boyd et. al 2005) as recalled below:

\[ P_{a,b}(x, y) = x^{\max(a-b,0)} \left( \frac{x^a - 1}{x - 1} + \frac{x^b - 1}{x - 1} y + x^{b-a} \frac{x^a - 1}{x - 1} y^2 \right) \]

\[ Q_{a,b}(x, y) = x^{\max(a-b,0)} (1 + x^a + (1 + x^b)y + x^{b-a}(1 + x^a)y^2) \]

where \( a + b \) is odd.

\[ R_{a,b}(x, y) = x^{\max(a-b,0)} (1 + x^a + (1 - x^b)y - x^{b-a}(1 + x^a)y^2) \]

where \( a \) and \( b \) are both odd.

\[ S_{a,b,\varepsilon}(x, y) = 1 + (x^a + \varepsilon)(x^b + \varepsilon)y + x^{a+b}y^2 \]

where \( 1 \leq a < b \), \( \gcd(a, b) = 1 \), and \( \varepsilon = \pm 1 \).

\begin{table}
\begin{tabular}{|c|c|c|}
\hline
Measure & Original polynomial & Low-Degree Polynomials \\
\hline
1.255433866 & \( P(2,3) \) & 1 - \( y^2 - x - xy^3 \)  
- \( x^2y \)  
+ \( x^3y^3 \)  
\hline
1.285734864 & \( P(1,3) \) or \( P(2,1) \) & \( y^2 - y^3 - x + xy^3 \)  
+ \( x^2 \)  
- \( x^2y \)  
\hline
1.315692702 & \( P(3,5) \) & \( y^3 + y^4 + x^2y^5 + x^3y^3 \)  
+ \( x^5y^4 \)  
+ \( x^5y^5 \)  
\hline
1.324717957 & \( T(1 + x - x^3) \) & 1 - \( y + y^3 - x + xy \)  
- \( xy^3 \)  
\hline
1.325372497 & \( P(3,4) \) & \( y^2 - y^3 - xy^4 + x^3y^2 \)  
+ \( x^4y^3 \)  
- \( x^4y^4 \)  
\hline
1.332051105 & \( P(2,5) \) & \( y^3 - y^4 - x^2y^3 + x^3y^5 \)  
+ \( x^5y^4 \)  
- \( x^5y^5 \)  
\hline
\end{tabular}
\end{table}
<table>
<thead>
<tr>
<th>Measure</th>
<th>Original polynomial</th>
<th>Low-Degree Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.332396129</td>
<td>$S$</td>
<td>$y - y^2 - xy^2 - x^3 y^2 - x^4 y^2 + x^4 y^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.338137431</td>
<td>$P(3,2)$</td>
<td>$y^4 - x - xy^2 + x^3 y^2 + x^3 y^4 - x^4$</td>
</tr>
<tr>
<td>1.340506882</td>
<td>$P(3,1)$</td>
<td>$1 + y^2 - xy - x^2 y + x^3 + x^3 y^2$</td>
</tr>
<tr>
<td>1.350316979</td>
<td>$S(1,4, -)$</td>
<td>$1 + y + xy - x^4 y^4 - x^5 y^4 - x^5 y^5$</td>
</tr>
<tr>
<td>1.351145895</td>
<td>$P(4,5)$</td>
<td>$y^4 - y^5 - xy^3 + x^4 y^5 + x^5 y^3 - x^5 y^4$</td>
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<tr>
<td>1.356748105</td>
<td>$P(4,3)$</td>
<td>$y + x + xy - x^4 y^4 - x^4 y^5 - x^5 y^4$</td>
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<tr>
<td>1.358545590</td>
<td>$P(4,1)$</td>
<td>$y^4 - x + x^2 y^2 + x^3 y^2 - x^4 y^4 + x^5$</td>
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<tr>
<td>1.364199545</td>
<td>$T(1 - x^2 + x^5)$</td>
<td>$y^2 - y^5 - x^2 y^2 - x^3 y^5 - x^5 y^2 + x^5 y^5$</td>
</tr>
<tr>
<td>1.364435811</td>
<td></td>
<td>$y^3 - x - xy + x^2 y^2 + x^2 y^3 - x^3$</td>
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<tr>
<td>1.366883070</td>
<td>$R(1,5)$</td>
<td>$y^5 - x^2 y^2 + x^2 y^4 - x^3 y + x^3 y^3 - x^5$</td>
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<tr>
<td>1.348651990</td>
<td>$y^3 + x^3 y^2 - y^2 + x^4 y - xy + x^4$</td>
<td>$y^2 - x - xy^3 + x^2 + x^2 y^3 - x^3 y$</td>
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<tr>
<td>1.361956458</td>
<td>$xy^4 - x^3 y^3 + xy^3 - x^2 y + y - x^2$</td>
<td>$y^3 - x - xy - x^2 y^3 - x^2 y^4 + x^3 y$</td>
</tr>
<tr>
<td>1.365269546</td>
<td>$-y^2 - x - xy^4 - x^3 y - x^3 y^5 - x^4 y^3$</td>
<td>$y - x - xy^4 + x^3 y + x^3 y^5 - x^4 y^4$</td>
</tr>
<tr>
<td>Measure</td>
<td>Original polynomial</td>
<td>Low-Degree Polynomials</td>
</tr>
<tr>
<td>------------------</td>
<td>-------------------------------------------------------------------------------------</td>
<td>----------------------------------------</td>
</tr>
<tr>
<td>1.366299075</td>
<td>$xy^3 + y^3 - x^4y^2 - xy + x^5 + x^4$</td>
<td>$xy^3 + y^3 - x^4y^2 - xy + x^5 + x^4$</td>
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<tr>
<td>1.366401994</td>
<td>$-1 + y + x^2y^4 - x^3y - x^5y^4 + x^5y^5$</td>
<td>$y^2 + y^3 - x^2y^4 - x^3 + x^5y$</td>
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<tr>
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<td>$x^3y^4 - x^3y^3 + x^2 + xy^4 - y + 1$</td>
<td>$y^3 - y^4 - x - x^2y^4 + x^3y$</td>
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<tr>
<td>1.367511030</td>
<td>$xy^3 + x^4y^2 + y^2 + x^4y + y + x^3$</td>
<td>$y^2 + y^3 - xy^4 - x^3y + x^4y^2 + x^4y^3$</td>
</tr>
<tr>
<td>1.368132226</td>
<td>$x^3y^3 - y^3 - x^3y^2 - xy - x^4 + x$</td>
<td>$y^4 - xy - xy^2 + x^3y^3 + x^3y^4 - x^4y$</td>
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<tr>
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<td>$x^2y^5 + xy^3 - y^3 - x^3y^2 + x^2y^2 + x$</td>
<td>$y - y^3 - xy^2 + x^3y + x^4$</td>
</tr>
<tr>
<td>1.368978737</td>
<td>$y^5 - x^2y^4 - x^3y^3 + y^2 + xy - x^3$</td>
<td>$y - y^2 - xy^4 - x^3y + x^4y^3 - x^4y^3$</td>
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<tr>
<td>1.369489377</td>
<td>$x^3y^4 - x^3y^3 - y^3 + x^5y + x^2y - x^2$</td>
<td>$y^3 + xy^3 - xy^5 + x^4 - x^4y^2 - x^5y^2$</td>
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<tr>
<td>1.359375641</td>
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<td>$y - y^3 - xy^4 - x^2 + x^3y + x^3y^3$</td>
</tr>
<tr>
<td>1.368922213</td>
<td>$y - y^3 - x^2 - x^3y^4 - x^5y + x^5y^3$</td>
<td>$y - y^3 - x^2 - x^3y^4 + x^5y - x^5y^3$</td>
</tr>
</tbody>
</table>

5. **CONCLUSION**

In this work, we have demonstrated that it is possible to lower the degree of bivariate polynomials known in the literature to validate the known limit points of Mahler’s measure of univariate polynomials. Some trials conducted at degree 6 suggest that a systematic study at (low) subsequent degrees could identify the majority of the known limit points, and potentially discover new ones.
REFERENCES