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WALKING DIAGONALLY: A SIMPLE PROOF OF COUNTABILITY OF THE SET OF ALL FINITE SUBSETS OF NATURALS

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Abstract

The countability of the set of finite subsets of natural numbers is derived. In addition to the derivation itself, the use of diagonal method is illustratively presented; thinking about infinite sets relate to the name of Georg Ferdinand Ludwig Philipp Cantor, the creator of set theory, which has become a fundamental theory in mathematics. The proof of the claim is original, although the theorem is an analogy with the famous theorem about the countability of rational numbers. A certain insight into the subsets of natural numbers is given. Anyway, the technique could be considered already at high school level as it is essential for mathematical thinking.

Keywords

set theory, diagonalization method, countability, teaching methods, university basic mathematics

1. INTRODUCTION

After the literary introduction in which the Czech writer described crawling along the diagonal of the square, we will go through the diagonal as well. Our article focused on a use of diagonals for certain infinite sets and is intended mainly didactically.

Proof strategies are important in mathematics as well as in mathematical education. We recall for example mathematical induction, proof by contradiction, proof by exhaustion, the Invariance Principle, Cantor's diagonal argument, the box principle but also very simple ideas like "use an adequate substitution" or "add zero in an appropriate way." (Stender and Stuhlmann, 2018)

Infinity is a popular topic of general interest and high school students, non-mathematics students and also educated public are certainly capable of appreciating a well-constructed argument, even if they have little interest in numbers. Cantor's diagonal proof (Cantor, 1891) that the set of real numbers is uncountable is one of the most famous arguments in modern mathematics and thinking as such. There are interesting approaches to familiarize this proof to students of the fine arts or humanities (e.g., also music, theatre, and literature) (Rauff, 2008).

The Cantor's diagonal argument is known especially for the case of proof of uncountability of reals. As to opposite, in a certain sense, rationals are countable. We will show a less familiar situation and we will target that the reader understands the generality of the method. Let **N** be a set of natural numbers. By Georg Cantor, the founder of the set theory, the cardinality of **N** was denoted by $\aleph_0$. Now, we denote the power set of **N** by $2^{\mathbf{N}}$. Evidently, it is not a set of lesser cardinality!

Moreover, it is not difficult to prove that there is no bijection $f: \mathbf{N} \to 2^{\mathbf{N}}$. So, let f be such a bijection and let $D = \{n; n \notin f(n)\}$. Then $D \in 2^{\mathbf{N}}$ and since f is a bijection, there exists $m \in \mathbf{N}$ such that $f(m) = D$. Now, note that $m \in D$ by the definition if and only if $m \notin f(m)$, so $m \in D$ if and only if $m \notin D$ and this is the contradiction.

The cardinality of $2^{\mathbf{N}}$ was denoted by Cantor as $\aleph_1$. Nevertheless, the set of rationals $\mathbf{Q}$ has the same cardinality as **N**.

In proof of this finding, we meet the diagonal argument, which we will use later. But our goal will be different than any investigation of rational numbers.

2. THE PROBLEM AND THE PROOF

For $k = 0,1,2, \ldots$, let F_k denote the set of all k-element subsets of **N** and let $F = \bigcup_k F_k$.

The (mathematical) purpose of this paper is to present an original way to proof that card $F = \aleph_0$ which can provide an interesting look at this and similar problems. This can be understood as an illustrative example of the method for students.

We note that we do not need a use of the axiom of choice or any of its equivalents.

PROPOSITION 1. *For every k,* card $F_k = \aleph_0$ *and therefore F_k is well-ordered.*

Proof. The assertion is trivial for $k = 0$. The assertion is trivial for $k = 1$, too. However, we present how F_1 is well ordered and how its elements are denoted:

$$F_1 = \{\{1\} \prec \{2\} \prec \{3\} \prec \cdots\}$$

This can be rewritten as:

$$F_1 = \{S_{1,1} \prec S_{1,2} \prec S_{1,3} \prec \cdots\}$$

For F_2, we introduce a little formalism. For an ordered couple (S, b), $S = \{a\} \in F_1$, $b \in \mathbf{N}$, we define its *associated set* T as $T = \{a\} \cup \{b\}$ and it is clear that T is either an element of F_1 (for $a = b$) or an element of F_2 (for $a \neq b$). Now, we introduce the order on $F_1 \times \mathbf{N}$ by the following way. We arrange the table:

$$\begin{tikzcd}
(\{1\},1) & (\{1\},2) & (\{1\},3) & \cdots \\
(\{2\},1) \arrow[ur] & (\{2\},2) \arrow[ur] & (\{2\},3) & \cdots \\
(\{3\},1) \arrow[ur] & (\{3\},2) & (\{3\},3) & \cdots \\
\cdots & & &
\end{tikzcd}$$

Now, we can order $F_1 \times \mathbf{N}$ by reading diagonals beginning from the top left corner. Moreover, we can even order F_2 by considering of corresponding associated sets under such rules that if an associated set is singleton or already used, we do not add it. So, we obtain:

$$F_2 = \{\{1,2\} \prec \{1,3\} \prec \{1,4\} \prec \{2,3\} \prec \{1,5\} \dots\}$$

Analogously, we continue with $F_2 \times \mathbf{N}$ and F_3:

$$\begin{tikzcd}
(\{1,2\},1) & (\{1,2\},2) & (\{1,2\},3) & \cdots \\
(\{1,3\},1) \arrow[ur] & (\{1,3\},2) \arrow[ur] & (\{1,3\},3) & \cdots \\
(\{1,4\},1) \arrow[ur] & (\{1,4\},2) & (\{1,4\},3) & \cdots \\
\cdots & & &
\end{tikzcd}$$

$$F_3 = \{\{1,2,3\} \prec \{1,2,4\} \prec \{1,3,4\} \prec \{1,2,5\} \prec \{1,3,5\} \dots\}$$

Similarly, for an arbitrary k. ■

REMARK 1. We see that we have created nothing but a graded order on F_k which is neither graded lexicographical order nor graded reverse lexicographical order. Indeed, there is the same total order on diagonals, however, we have derived: $\{1,3,4\} \prec \{1,2,5\}$

PROPOSITION 2. card $F = \aleph_0$ *and therefore whole* F *is well-ordered.*

Proof. We already have all F_k well-ordered. So, we can repeat the process.

$$\begin{tikzcd}
S_{1,1} & S_{1,2} & S_{1,3} & \cdots \\
S_{2,1} \arrow[ur] & S_{2,2} \arrow[ur] & S_{2,3} & \cdots \\
S_{3,1} \arrow[ur] & S_{3,2} & S_{3,3} & \cdots \\
\cdots & & &
\end{tikzcd}$$

Starting with the empty set a reading of diagonals provides

$$F = \{\emptyset \prec S_{1,1} \prec S_{2,1} \prec S_{1,2} \prec S_{3,1} \prec S_{2,2} \dots\}$$

or

$$F = \{\emptyset \prec \{1\} \prec \{1,2\} \prec \{2\} \prec \{1,2,3\} \prec \{1,3\} \dots\}$$

■

REMARK 2. The result is widely known for a long time, and it is a special case of a more general result which states that the set of all finite subsets of a countable set is countable. Another simple ordering of finite sets may be this one: after the empty set we start with a singleton $\{1\}$ and then gradually add a following singletons together with all unions with sets previously added, i. e.

$$\emptyset \prec \{1\} \prec \{2\} \prec \{1,2\} \prec \{3\} \prec \{1,3\} \prec \{2,3\} \prec \{1,2,3\} \prec \{4\} \dots$$

in which we do not need to use a diagonalization.

3. ON A GENERALIZATION

The following consideration is not fully formalized, as it is nothing but a suggestion for certain ideas which can be elaborated by students.

Let $F(R)$ be a set of infinite subsets of $\mathbf{N}$ emerging under a certain rule R from sets of F. For instance, let R works by assigning of the set

$$\dot{A} = A \cup \{m+1, m+2, m+3, \dots\}$$

where m is a maximal element in $A \in F$. ($\dot{\emptyset} = \mathbf{N}$.) Then $F(R)$ contains only infinite subsets, however, the cardinality of them does not exceed $\aleph_0$. So, when we wonder where the enormous power of the new $\aleph_1$ is hidden, it is in such infinite subsets of **N** for which there is no rule how to get them from finite subsets of **N**.

The Cantor's diagonalization argument is based on the following. Let us consider objects described by infinite sequences $(c_1, c_2, \dots)$, where c_k reach different values, so at least two values, say α and β. Then the collection of all such objects of the type $\alpha\alpha\beta\beta\beta\alpha\beta\alpha\alpha\alpha\beta\beta\alpha\alpha\beta\beta\alpha\beta\alpha\beta\alpha$ ... is nondenumerable: if we suppose that it is possible, then we constuct new object having with *k*-th object different values on the *k*-th position and see by this that the list is not complete. If we have infinite subsets of **N** with "no rule", the only way how to describe them is by lists of their elements, so equivalently, we set $c_k = \alpha$ if *k* belongs to such a subset and $c_k = \beta$ if not.

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