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A POISSON-AKASH INRAR(1) MODEL FOR OVER DISPERSED COUNT TIME SERIES

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Abstract

In this paper, a Poisson-Akash INAR(1) model was proposed in order to improve on the modelling of overdispersed stationary count time series. The estimators of the parameters of the model were derived using the Yule-Walker (YW) method and the conditional least squares (CLS) method. An expression for the conditional log-likelihood and the r-step ahead forecast were obtained for the model. Three overdispersed nonseasonal stationary count time series were modelled to illustrate the applicability of the proposed model as well as its capacity to outperform the competing INAR (1) models in modelling overdispersed stationary count time series and the result showed that the proposed model is a strong competitor in the analysis of overdispersed stationary count time series and can perform better than the competing INAR(1) models for some data sets.

Keywords

Count time series, over-dispersion, stationary, binomial thinning, and Poisson-Akash innovation

1. INTRODUCTION

A time series $\{X_t\}_{t=0}^{T}$ is called a count time series if $X_t \in \mathrm{N}_0 \ \forall t$, where t and N_0 are the time index and set of counting numbers respectively. Count time series are observed in several areas of life such as medicine (monthly number of persons suffering from a certain disease), business (monthly number of stock exchange), insurance (monthly number of claims on a given scheme), academics (yearly number of students admitted) among others.

Recently, serious attention has been given to the analysis of time dependent count data and with deepening of research, a new class of models called integer-valued autoregressive (INAR) models has been constructed based on thinning operators, specifically for the analysis and interpretation of count time series. The rationale behind the construction of INAR models is that the standard time series models such as ARMA models assume Gaussian error term (innovation) which may not be appropriate for count data modelling.

The pioneer work on INAR models is the INAR(1) model independently given by Mckenzie (1985) and Al-Osh and Alzaid (1987). An INAR(1) model is of the form

$$X_t = \alpha \circ X_{t-1} + I_t, \qquad (1)$$

where $\alpha \in (0,1)$ and I_t ,called the innovation of the process, is a non-negative random variable with mean μ_I and variance σ_I^2 .

In Eq. (1), the first term, the term involving the operator is defined as Steutel and van Harn (1979)

$$\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_{i,t-1} \qquad (2)$$

where $Y_{i,t-1}$ are Bernoulli random variables such that $P(Y_{i,t-1} = 1) = 1 - P(Y_{i,t-1} = 0) = \alpha$. Notably, the presence of the operator in Eq. (1) is responsible for nonlinearity of the INAR model (Jung, 2006).

Eq. (1) is a convolution of two random components $\alpha \circ X_{t-1}$ and I_t, and employing the properties of binomial thinning operator given by Silva and Oliveira (2004) and Weiß (2008) made the model easily practicable. The binomial thinning operator is the first thinning operator used in the construction of INAR models. Some other thinning operators have also been used. Kim and Park (2008) proposed an extension of INAR(1) model using signed binomial thinning operator The negative binomial thinning operator was used to construct INAR models by Restic′ *et al*. (2009), Nastic′ *et al*. (2012) and Tian *et al*. (2018), and mixed binomial thinning operator was used by Nastic′ *et al* (2017). The choice of the thinning operator to employ in the construction of an INAR model depends on the underlying process that generated the count time series at hand (Nastic′ *et al*, 2017).

Another crucial choice to make when constructing an INAR model is the choice of innovation distribution. This choice of innovation to employ depends on the dispersion structure of the count time series to be analysed. Mckenzie (1985) and Al-Osh and Alzaid (1987) assumed Poisson innovation (I_t). However, Poisson innovation is suitable for the application of INAR model to equidispersed count time series (a count time series that has equal mean and variance). Sometimes, a count time series may be overdispersed (when its variance exceeds the mean) or underdispersed (when its mean exceeds the variance). Aghababaei *et al*. (2012) have shown that any self-decomposable discrete distribution can serve as the innovation of an INAR model and however pointed out that overdispersed count time series requires overdispersed innovation and underdispersed count time series requires underdispersed innovation. Some works have been done to address the issue of overdispersion in count time series (Mckenzie ,1986; Alzaid and Al-Osh , 1988; Aghababaei *et al*., 2012; Ferland *et al*., 2006; Tian *et al*., 2018; Mohammadpour *et al*. (2018); Tito *et al*., 2018; Okereke *et al*., 2019).

Mohammadpour *et al*. (2018) noted that sometimes the INAR models, despite the modifications done in the area of thinning operator and innovation in order to improve analysis, produce less accurate results. They suggested making a shift from the use of some traditional distributions such as Poisson, geometric and negative binomial distributions as the innovation distribution for INAR models to some mixture distributions and thus proposed a Poisson-Lindley INAR(1) (PLINAR(1)) model for the analysis of overdispersed stationary count time series. The PLINAR(1) model was found to be a more flexible model for analysis of count time series than other INAR(1) models constructed with innovations that follow some traditional distributions. Tito *et al*. (2018) emphasized that the innovation structure of PLINAR(1) model is complex and that its conditional probability function has no closed form. Hence, they proposed an INAR(1) model with Poisson-Lindley innovation (INARPL(1)) that has a simpler innovation structure and closed form conditional probabilities. The INARPL(1) model was found to be better than the PLINAR(1), the Poisson INAR(1) and the negative binomial INAR(1) models in the analysis of the count data on weekly sales of Zest white water 15oz soap.

This paper proposes a Poisson-Akash INAR(1) model (PAINAR(1)) based on the binomial thinning operator for the analysis of overdispersed count time series and compares its performance with the Poisson INAR(1) (PINAR(1)) model of Mckenzie (1985) and Al-Osh and Alzaid (1987), the geometric INAR(1 (INARG(1))) model of Aghababaei *et al*. (2012), the negative binomial INAR(1) (NBIINAR(1)) model of Al-Osh and Aly (1992), the Poisson-Lindley INAR(1) (PLINAR(1)) model of Mohammadpour *et al*. (2018) and the Poisson-Lindley INAR(1) (INARPL(1)) model given by Tito *et al*. (2018). The paper was motivated generally by the quest to continue improving on the analysis of count time series using INAR models and specifically by the fact that Shanker (2017) proposed the Poisson-Akash distribution, as a Poisson mixture of the Akash distribution of Shanker (2015), and found that it can be a better distribution than the Poisson-Lindley distribution which Mohammadpour *et al*. (2018) and Tito *et al*. (2018) used as innovation in PLINAR(1) model and INARPL(1) model respectively. The proposed PAINAR(1) model and its properties are given in Section 2, Section 3 shows the application of the models on real data and the study is concluded in Section 4.

2. THE PROPOSED PAINAR(1) MODEL AND ITS PROPERTIES

Definition 1 (Shanker, 2017): A discrete random variable X is said to have Poisson-Akash distribution with parameter $\theta\ (PA(\theta))$ if its probability mass function (pmf) is given as

$$P(X=k)=\frac{\theta^3}{\theta^2+2}.\frac{k^2+3k+\left(\theta^2+2\theta+3\right)}{(\theta+1)^{k+3}},\quad k=0,1,2,......,\theta>0, \tag{3}$$

Eq. (3) is a Poisson mixture of the Akash distribution of Shanker (2015).

The first moment about the origin, the first central moment and the probability generating function (pgf) of a Poisson-Akash random variable are respectively given as

$$\mu=\frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\quad , \tag{4}$$

$$\sigma^2=\frac{\theta^5+\theta^4+8\theta^3+16\theta^2+12\theta+12}{\theta^2\left(\theta^2+2\right)^2} \tag{5}$$

and

$$P_k(t)=\frac{\theta^3}{\left(\theta^2+2\right)(\theta+1)^2}\left[\frac{\theta+2\theta+3}{(\theta+1-t)}+\frac{2t(3(\theta+1)-2t)}{(\theta+1-t)^3}\right] \tag{6}$$

According to Shanker (2017), the relationship between Eq. (4) and Eq. (5) is of the form

$$\sigma^2 = \mu\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]$$

which implies that the distribution in Eq. (3) is overdispersed.

Let the innovation I_t in Eq. (1) follow Poisson-Akash distribution with parameter θ. Then Eq. (1) becomes a Poisson-Akash INAR(1) model (PAINAR(1)). The Poisson-Akash INAR(1) (PAINAR(1)) model is constructed for the analysis of overdispersed count time series

Proposition 1: Let I_t be an i.i.d sequence of Poisson-Akash distributed random variables with parameter θ. If $\alpha \in (0,1)$ and X_t is a stationary PAINAR(1) model. Then the unconditional mean and the unconditional variance of X_t are respectively given as

$$\text{(a) } E(X_t) = \frac{\theta^2+6}{(1-\alpha)(\theta^2+2)\theta}\ , \tag{7}$$

$$\text{(b) } Var(X_t) = \frac{\theta^2+6}{(1-\alpha^2)(\theta^2+2)\theta}\left[1+\alpha+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right] \tag{8}$$

See proof in Appendix 1

Eq. (1) can be recursively written in terms of the innovation as

$$X_t = \alpha^k \circ X_{t-k} + \sum_{j=0}^{k-1}\alpha^j \circ I_{t-j} \tag{9}$$

Eq. (9) implies self-decomposability of a PAINAR(1) model. Notably, for

i. $\alpha \in [0,1)$, the PAINAR(1) is second- order stationary with autocovariance function $Cov(X_t, X_{t+k}) = \sigma^2\alpha^k$

ii. $\alpha = 1$, the PAINAR(1) is nonstationary.

However, this study is restricted to the case in (i)

Eq. (10) is the probability of transiting from state i to state j in one step.

$$P(X_t = j | X_{t-1} = i) = P(\alpha \circ X_{t-1} + I_t = j | X_{t-1} = i).$$

$$= \sum_{k=0}^{\min(i,j)} P(\alpha \circ X_{t-1} = k | X_{t-1} = i) P(I_t = j-k)$$

$$= \sum_{k=0}^{\min(i,j)} \binom{i}{k} \alpha^k (1-\alpha)^{i-k} \frac{\theta^3}{\theta^2+2} \cdot \frac{(j-k)^2 + 3(j-k) + (\theta^2+2\theta+3)}{(\theta+1)^{j-k+3}},, \tag{10}$$

for $i, j = 0,1,\ldots.$

The PAINAR(1) model is indeed a Markov process.

2.1 Estimation

Three methods of estimation are considered in this study. The methods are the Yule-Walker (YW), the conditional least squares (CLS) and the conditional maximum likelihood (CML) methods of estimation.

2.1.1 Yule-Walker estimation

The Yule-walker estimator for the parameter α of PAINAR(1) model can be obtained from the autocovariance function as

$$\hat{\alpha}_{YW} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = \frac{\sum_{t=2}^{T}\left(X_t - \overline{X}\right)\left(X_{t-1} - \overline{X}\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2} \tag{11}$$

Equating Eq. (7) to its corresponding sample moment leads to the equation

$$\frac{\theta^2 + 6}{(1-\alpha)\left(\theta^2 + 2\right)\theta} = \overline{X}$$

$$\Rightarrow a\theta^3 - \theta^2 + 2a\theta - 6 = 0 \tag{12}$$

where $a = \left(1 - \hat{\alpha}_{YW}\right)\overline{X}$ and $\overline{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$.

Consequently, we solve the cubic equation in Eq. (12) to find the YW estimator of the parameter θ of PAINAR(1) model.

2.1.2 Conditional least squares estimation

The CLS estimator $\hat{\phi} = \left(\hat{\alpha}_{CLS}, \hat{\theta}_{CLS}\right)^I$ of the parameter vector $\phi = \left(\alpha_{CLS}, \theta_{CLS}\right)^I$ of PAINAR(1) model is given by

$$\hat{\phi} = \arg \min_{T} \left(S_T(\phi)\right)$$

where $S_T(\phi) = \sum\left(X_t - E\left(X_t \middle| X_{t-1}\right)\right)^2$. Let $\hat{\alpha}_{CLS}$ and $\hat{\theta}_{CLS}$ be the CLS estimators of α and θ respectively. Then

$$\hat{\alpha}_{CLS} = \frac{(T-1)\sum_{t=2}^{T} X_t X_{t-1} - \sum_{t=2}^{T} X_t \sum_{t=2}^{T} X_{t-1}}{(T-1)\sum_{t=2}^{T} X_{t-1}^2 - \left(\sum_{t=2}^{T} X_{t-1}\right)^2} \tag{13}$$

and

$$\frac{\theta^2 + 6}{\left(\theta^2 + 2\right)\theta} - = \frac{\sum_{t=2}^{T} X_t}{T-1} - \hat{\alpha}_{CLS}\frac{\sum_{t=2}^{T} X_{t-1}}{T-1}$$

$$\Rightarrow A\theta^3 - (T-1)\theta^2 + 2A\theta - (T-1)6 = 0 \tag{14}$$

where $A = \sum_{t=2}^{T} X_t - \hat{\alpha}_{CLS} \sum_{t=2}^{T} X_{t-1}$

Again, we solve for θ in Eq. (14) so as to obtain the CLS estimator of θ. Several authors have discussed the techniques of solving cubic equations (Cardano, 1545; Ostrowski, 1966; Mukundan, 2010; Okereke *et al*., 2014). Given the values of A and T, RConics, which is one of R packages is used to solve Eq. (12) and Eq. (14) in order to obtain the YW and the CLS estimates of θ in Section 3.

2.1.3 Conditional maximum likelihood estimation

The conditional probability distribution of X_t given X_{t-1} for PAINAR(1) model following Eq. (10) is a convolution of binomial distribution $b(X_{t-1}, \alpha)$ and Poisson-Akash distribution $PA(\theta)$. Hence, the conditional likelihood function is

$$L(\alpha,\theta) = \prod_{t=11}^{T} P\left(X_t = x_t \middle| X_{t-1} = x_{t-1}\right)$$

$$= \prod_{t=1}^{T} \frac{\theta^3}{\theta^2+2} \sum_{k=0}^{\min(x_t, x_{t-1})} \binom{x_{t-1}}{ki} \alpha^k (1-\alpha)^{X_{t-1}-k} \frac{(x_t-k)^2 + 3(x_t-k) + \left(\theta^2+2\theta+3\right)}{(\theta+1)^{x_t-k+3}},$$

and the log-likelihood function is

$$\ell(\alpha,\theta) = \sum_{t=1}^{T} \log\left[\frac{\theta^3}{\theta^2+2} \sum_{k=0}^{\min(x_t, x_{t-1})} \binom{x_{t-1}}{k} \alpha^k (1-\alpha)^{X_{t-1}-k} \frac{(x_t-k)^2 + 3(x_t-k) + \left(\theta^2+2\theta+3\right)}{(\theta+1)^{x_t-k+3}}\right]$$

The CML estimate of $\phi = (\alpha_{CML}, \theta_{CML})^I$ is the value $\hat{\phi} = \left(\hat{\alpha}_{CML}, \hat{\theta}_{CML}\right)^I$ that maximizes log-likelihood function. Obviously, there is no close form for $\hat{\phi} = \left(\hat{\alpha}_{CML}, \hat{\theta}_{CML}\right)^I$ of PAINAR(1) model just like every other INAR model and hence the estimates of $\phi = (\alpha_{CML}, \theta_{CML})^I$ may be obtained numerically.

2.2 Forecast

Proposition 2: Let X_t be a stationary PAINAR(1) model and $T, r \in \mathrm{N}$. Then, the minimum mean square error (MMSE) *r*-step ahead forecast of X_{T+r} have the following properties

(a) $$E\left(X_{T+r} \middle| \delta_T\right) = \alpha^r X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{1-\alpha^r}{1-\alpha}\right] \tag{15}$$

(b) $$Var\left(X_{T+r} \middle| \delta_T\right) = \alpha^r\left(1-\alpha^r\right)X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{1-\alpha^r}{1-\alpha}\right] + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]\left[\frac{1-\alpha^{2r}}{1-\alpha^2}\right] \tag{16}$$

(c) $$\lim_{r\to\infty} E\left(X_{T+r}\middle|\delta_T\right) = E\left(X_t\right) \tag{17}$$

(d) $$\lim_{r\to\infty} Var\left(X_{T+r}\middle|\delta_T\right) = Var\left(X_t\right) \tag{18}$$

See proof in Appendix 2.

3. REAL DATA ANALYSIS

Three real data sets are analysed in this section in order to demonstrate the application of PAINAR(1) model. The first two of the data sets are data sets on the monthly number of submissions to animal health from 2003 to 2009, earlier used by Aghababaei *et al.* (2012) and Mohammadpour *et al.* (2018), and the third data set is on the weekly sales of Zest White Water 15oz soap analysed by Tito *et al.* (2018). The first set of data is the skin lesions data with $\text{mean} = 1.43$, $\text{variance} = 3.36$, $\text{skewness} = 1.87$ and $\text{kurtosis} = 7.06$.The second set of data is the anorexia data with $\text{mean} = 0.82$, $\text{variance} = 2.90$ skewness $= 3.38$ and $\text{kurtosis} = 17.7$. Mohammadpour *et al.* (2018) gave the dispersion index of the data sets and also carried out a formal test to determine the dispersion structure of the data. They concluded (at 5% significance level) that the data sets are overdispersed. Aghababaei *et al.* (2012) and Mohammadpour *et al.* (2018) have shown that an INAR(1) model is suitable for analysis of the two data sets. The third data set, the weekly sales of soap has 242 observations with mean = 5.44, variance = 15.40, skewness = 1.51 and kurtosis = 5.88 and is obtainable from Kits Center for Marketing, Graduate School of Business of the University of Chicago via http://gbswww.uchicago.edu/kilts/research/db/dominicks. Tito *et al.* (2018) found that the third data are overdispersed at 1% significant level and also inferred that a first order autoregressive process is suitable for the data. Hence we proceed to analyse the data sets using the PAINAR(1) model.

It is well known that the CML estimate of an INAR model performs better than the YW and the CLS estimates (Marcelo and Klaus, 2015; Marcelo *et al.*, 2015). Therefore, we compare the PAINAR(1) model with the PINAR(1) model, the INARG(1) model, the NBIINAR(1) model, the PLINAR(1) model and the INARPL(1) model on the basis of the CML estimates for the three data sets. The Optim function in R was used in maximizing the log-likelihood (LL) function in Subsection 2.1.3 to obtain the CML estimates using the YW estimates, as obtained using Eq. (11) and Eq. (12), as the initial values. From the CML estimates obtained, the Akaike information criterion (AIC), the Bayesian information criterion (BIC) and the root mean squared error (RMS) were evaluated for the three sets of data and the corresponding results are as shown in Table 1, Table 2 and Table 3 respectively.

Table 1: Estimates of Parameter of INAR Models Fitted to the Skin Lesion Data and their Corresponding Model Selection Criteria

Model	CML Estimate	LL	AIC	BIC	RMS
PINAR(1)	$\hat{\alpha} = 0.173$ $\hat{\lambda} = 1.172$	-151.11	306.22	311.08	1.78
INARG(1)	$\hat{\alpha} = 0.118$ $\hat{\pi} = 0.444$	-134,96	273.91	278.47	1.76
NBIINAR(1)	$\hat{n} = 1.01$ $\hat{p} = 1.03$ $\hat{\rho} = 0.31$	-135.07	276.14	281.00	1.78.
PLINAR(1)	$\hat{\alpha} = 0.246$ $\hat{\theta} = 1.054$	-109.91	223.81	228.67	1.42
INARPL(1)	$\hat{\alpha} = 0.112$ $\hat{\theta} = 1.165$	-135.3743	274.75	279.30	1.77
PAINAR(1)	$\hat{\alpha} = 0.116$ $\hat{\theta} = 1.543$	-135.3707	274.74	279.30	1.77

Table 1 shows that the PAINAR(1) model performed better than PINAR(1) model, NBIINAR(1) model and INARPL(1) model while INARG(1) model and PLINAR(1) model performed better than our model in the analysis of the Skin Lesion data.

Table 2: Estimates of Parameters of INAR Models Fitted to the Anorexia Data and their Corresponding Model Selection Criteria

Model	CML Estimate	LL	AIC	BIC	RMS
PINAR(1)	$\hat{\alpha}=0.385$ $\hat{\lambda}=0.511$	-112.52	229.04	233.91	1.51
INARG(1)	$\hat{\alpha}=0.315$ $\hat{\pi}=0.637$	-95.55	195.10	199.65	1.50
NBIINAR(1)	$\hat{n}=0.26$ $\hat{p}=0.95$	-86.30	178.60	183.46	1.50.
PLINAR(1)	$\hat{\rho}=0.65$ $\hat{\alpha}=0.49$ $\hat{\theta}=1.71$	-85.45	174.19	179.05	1.48
INARPL(1)	$\hat{\alpha}=0.324$ $\hat{\theta}=2.349$	-97.90	199.80	204.35	1.50
PAINAR(1)	$\hat{\alpha}=0.323$ $\hat{\theta}=2.642$	-97.03	198.06	202.61	1.50

Observe from Table 2 that the PAINAR(1) performed better than PINAR(1) model, and INARPL(1) model while INARG(1) model, NBIINAR(1) model and PLINAR(1) model performed better than our model in the analysis of the Anorexia data.

Table 3: Estimates of Parameters of INAR Models Fitted to the Weekly Sales of Soap Data and their Corresponding Model Selection Criteria

Model	CML Estimate	LL	AIC	BIC	RMS
PINAR(1)	$\hat{\alpha}=0.2340$ $\hat{\lambda}=4.1855$	-679.905	1363.81	1370.79	3.65
PLINAR(1)	$\hat{\alpha}=0.3152$ $\hat{\theta}=0.3516$	-641.560	1287.12	1294.10	3.63
INARPL(1)	$\hat{\alpha}=0.3202$ $\hat{\theta}=0.4533$	-622.5360	1249.07	1256.05	3.61
PAINAR(1)	$\hat{\alpha}=0.2756$ $\hat{\theta}=0.6647$	-620.7479	1245.50	1252.47	3.60

Table 3 shows that the PAINAR(1) model performed better than PINAR(1) model, and PLINAR(1) model and INARPL(1) model in the analysis of the Weekly soap data.

Lastly, we compare the forecast accuracy of the models obtained using innovations that are mixed distribution. The comparison was done by using 232 observations out of the 242 observations of the weekly sales of soap to forecast the remaining 10 observations and hence evaluate the mean square error (MSE) of the r-step ahead forecast.

Table 4: r-Step Ahead Conditional Mean Forecast for Weekly Sales of Soap

r	X_{232+r}	Model		
		PLINAR(1)	INARPL(1)	PAINAR(1)
1	9	1.2608	1.2808	1.1024
2	4	7.7852	6.6058	6.4493
3	14	7.7690	6.1972	6.1651
4	2	11.4136	9.7811	9.2226
5	2	7.7852	6.0609	5.9985
6	1	7.8340	6.1000	6.0215
7	5	7.5323	5.7924	5.7521
8	2	8.7998	7.0772	6.8562
9	6	7.8557	6.1179	6.0299
10	4	9.1171	7.3991	7.1324
MSE		36.4106	26.8279	25.7085

Table 4 shows that the PAINAR(1) model produced the minimum forecast mean squared error (MMSE) and thus has a better forecast accuracy than the PLINAR(1) model and the INARPL(1) model in the forecast of the weekly sales of soap.

4. CONCLUSION

A new INAR(1) model named Poisson-Akash INAR(1) model (PAINAR(1)) has been proposed. Some basic properties of the model such as the unconditional mean and variance, the YW and the CLS estimators of the parameters of the model, the log likelihood function and the r-step ahead forecast were derived. Numerical results namely, conditional maximum likelihood estimates of the parameters of the new INAR model and values of some model selection criteria pertaining to three real data indicate that the rating of the performance of an INAR model is affected by the innovation distribution and dispersion characteristic of the count time series under consideration. Interestingly, there exists empirical evidence that the model introduced in this article can outperform well-known INAR(1) for overdispersed stationary count time series.

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Appendix 1: Proof of Proposition 1

Let the innovation I_t in Eq. (1), follow Poisson-Akash distribution with parameter θ such that the mean and the variance of I_t are $\mu_I = \frac{\theta^2+6}{\theta(\theta^2+2)}$ and $\sigma_I^2 = \frac{\theta^2+6}{\theta(\theta^2+2)}\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]$ respectively. Then the conditional mean and variance of X_t are given by

$$E(X_t|X_{t-1}) = E((\alpha \circ X_{t-1} + I_t)|X_{t-1})$$

$$= E(\alpha \circ X_{t-1}|X_{t-1}) + E(I_t|X_{t-1}) = \alpha X_{t-1} + \frac{\theta^2+6}{\theta(\theta^2+2)}$$

and

$$Var(X_t|X_{t-1}) = Var((\alpha \circ X_{t-1} + I_t)|X_{t-1}) = Var(\alpha \circ X_{t-1}|X_{t-1}) + Var(I_t|X_{t-1})$$

$$= \alpha(1-\alpha)X_{t-1} + \frac{\theta^2+6}{\theta(\theta^2+2)}\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]$$

respectively. Assuming stationarity, the marginal mean and variance of X_t are respectively given by

$$E(X_t) = E(E(X_t|X_{t-1})) = E\left(\alpha X_{t-1} + \frac{\theta^2+6}{\theta(\theta^2+2)}\right) = \alpha E(X_{t-1}) + \frac{\theta^2+6}{\theta(\theta^2+2)} = \alpha E(X_t) + \frac{\theta^2+6}{\theta(\theta^2+2)}$$

$$\Rightarrow E(X_t) = \frac{\theta^2+6}{(1-\alpha)(\theta^2+2)\theta}.$$

and

$$Var(X_t) = E(Var(X_t|X_{t-1})) + Var(E(X_t|X_{t-1}))$$

$$= E\left(\alpha(1-\alpha)X_{t-1} + \frac{\theta^2+6}{\theta(\theta^2+2)}\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]\right) + Var\left(\alpha X_{t-1} + \frac{\theta^2+6}{\theta(\theta^2+2)}\right)$$

$$= \alpha(1-\alpha)E(X_{t-1}) + \frac{\theta^2+6}{\theta(\theta^2+2)}\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right] + \alpha^2 Var(X_{t-1})$$

$$\Rightarrow (1-\alpha^2)Var(X_t) = \alpha\left[\frac{\theta^2+6}{\theta(\theta^2+2)}\right] + \frac{\theta^2+6}{\theta(\theta^2+2)}\left[1+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]$$

$$\Rightarrow Var(X_t) = \frac{\theta^2+6}{(1-\alpha^2)(\theta^2+2)\theta}\left[1+\alpha+\frac{\theta^4+16\theta^2+12}{\theta(\theta^2+2)(\theta^2+6)}\right]$$

Appendix 2: Proof of Proposition 2

From Eq. (1) *r*-step ahead forecast equation is given by

$$X_{T+r} = \alpha \circ X_{T+r-1} + I_{T+r}$$

Which can be represented recursively as

$$X_{T+r} \overset{d}{=} \alpha^r \circ X_T + \sum_{j=0}^{r-1} \alpha^j \circ I_{T+r-j}$$

For PAINAR(1) model $\alpha^r \circ X_T$ follow binomial distribution with parameters X_T and α^r while I_{T+r-j} follow Poisson-Akash with parameter θ. Using MMSE criterion the *r*-step ahead forecast of X_T is the conditional expectation of X_{T+r} given a σ-algebra δ_T. Thus the condition mean *r*-step forecast function is given by

$$\text{(a) } E(X_{T+r}|\delta_T) = E\left(\left(\alpha^r \circ X_T + \sum_{j=0}^{r-1} \alpha^j \circ I_{T+r-j}\right)\middle|\delta_T\right)$$

$$= E\left(\alpha^r \circ X_T \middle| \delta_T\right) + E\left(\sum_{j=0}^{r-1} \alpha^{j} \circ I_{T+r-j} \middle| \delta_T\right)$$

$$= \alpha^r X_T + \frac{\theta^2 + 6}{\theta\left(\theta^2 + 2\right)} \sum_{j=0}^{r-1} \alpha^j$$

$$= \alpha^r X_T + \frac{\theta^2 + 6}{\theta\left(\theta^2 + 2\right)} \left[\frac{1-\alpha^r}{1-\alpha}\right]$$

and the conditional variance of the r-step ahead forecast function as

$$\text{(b)}\, Var\left(X_T \middle| \delta_T\right) = Var\left(\left(\alpha^r \circ X_T + \sum_{j=0}^{r-1} \alpha^{j} \circ I_{T+r-j}\right) \middle| \delta_T\right)$$

$$= Var\left(\alpha^r \circ X_T \middle| \delta_T\right) + Var\left(\left(\sum_{j=0}^{r-1} \alpha^{j} \circ I_{T+r-j}\right) \middle| \delta_T\right)$$

$$= \alpha^r\left(1-\alpha^r\right)X_T + \sum_{j=0}^{r-1}\left[E\left(Var\left(\alpha^j \circ I_{T+r-j}\right)\right) + Var\left(E\left(\alpha^j \circ I_{T+r-j}\right)\right)\right]$$

$$= \alpha^r\left(1-\alpha^r\right)X_T + \sum_{j=0}^{r-1}\left[\alpha^j\left(1-\alpha^j\right)\frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\right]$$

$$+ \alpha^{2j}\left[\frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\right]\left[1 + \frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]$$

$$= \alpha^r\left(1-\alpha^r\right)X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\sum_{j=0}^{r-1}\left[\alpha^j + \alpha^{2j}\left[\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]\right]$$

$$= \alpha^r\left(1-\alpha^r\right)X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\sum_{j=0}^{r-1}\alpha^j + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]\sum_{j=0}^{r-1}\alpha^{2j}$$

$$= \alpha^r\left(1-\alpha^r\right)X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{1-\alpha^r}{1-\alpha}\right]$$

$$+ \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]\left[\frac{1-\alpha^{2r}}{1-\alpha^2}\right]$$

Taking the limit of the functions as $r \to \infty$, we have

$$\text{(c)}\, \lim_{r\to\infty} E\left(X_{T+r} \middle| \delta_T\right) = \lim_{r\to\infty}\left[\alpha^r X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{1-\alpha^r}{1-\alpha}\right]\right] = \frac{\theta^2+6}{\left(1-\alpha\right)\left(\theta^2+2\right)\theta} = E\left(X_t\right),$$

and

$$\text{(d)} \lim_{r\to\infty} Var\left(X_{T+r}\middle|\delta_T\right)= \lim_{r\to\infty}\left[\alpha^r\left(1-\alpha^r\right)X_T + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{1-\alpha^r}{1-\alpha}\right]\right.$$

$$\left. + \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}\left[\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]\left[\frac{1-\alpha^{2r}}{1-\alpha^2}\right]\right]$$

$$= \frac{\theta^2+6}{\left(1-\alpha^2\right)\left(\theta^2+2\right)}\left[1+\alpha+\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}\right]=Var\left(X_t\right).$$